TRIGONOMETRIE

585. Se consideră funcția $f(x) = \cos^{2n} x + \sin^{2n} x$, $n \in \mathbb{N}$, $n \geq 2$, $x \in \mathbb{R}$.

Mulțimea soluțiilor ecuației $f(x) = 1$?

Ⓡ Notăm $\sin^2 x = t \in [0,1]$ și introducem funcția

$$g(t) = (1-t)^n + t^n, \quad n \geq 2.$$

Atunci $g'(t) = -n(1-t)^{n-1} + n t^{n-1}$ și

$$g'(t) = 0 \iff \left(\frac{1}{t} - 1\right)^{n-1} = 1 \iff t = \frac{1}{2}.$$

Avem:

t	0		$\frac{1}{2}$		1
$g'(t)$	$- - -$		0	$+ + +$	
$g(t)$	1	$\searrow$	$\frac{1}{2^{n-1}}$	$\nearrow$	1

Deci $g(t) = 1 \iff t \in \{0, 1\} \implies$

$$f(x) = 1 \iff \sin^2 x \in \{0, 1\} \iff$$

$$x \in \{k\pi \mid k \in \mathbb{Z}\} \cup \left\{(2k+1)\frac{\pi}{2} \mid k \in \mathbb{Z}\right\} \iff$$

$$x \in \left\{k\frac{\pi}{2} \mid k \in \mathbb{Z}\right\}. \quad \boxed{D}$$

586. Mulțimea valorilor funcției f este:

$$g(t) \in \left[\frac{1}{2^{n-1}}, 1\right] \implies f(x) \in \left[\frac{1}{2^{n-1}}, 1\right] \quad \boxed{D}.$$

591-592. Fie x_1 și x_2 rădăcinile ecuației $x^2 - 2\sqrt{2}x + 1 = 0$.

$\operatorname{arctg} x_1 + \operatorname{arctg} x_2$ este:

R) Avem $x_1 = \sqrt{2} - 1$ și $x_2 = \sqrt{2} + 1$.

Deci $x_2 = \dfrac{1}{x_1}$.

Din formula $\operatorname{arctg} x + \operatorname{arctg}\dfrac{1}{x} = \dfrac{\pi}{2}$ deducem

$\operatorname{arctg} x_1 + \operatorname{arctg} x_2 = \dfrac{\pi}{2}$ $\boxed{A}$.

$\operatorname{arctg} x_1 \cdot \operatorname{arctg} x_2$ este:

Din formula $2\operatorname{arctg} x = \operatorname{arctg}\dfrac{2x}{1-x^2}$, $|x| < 1$

deducem:

$2\operatorname{arctg} x_1 = 2\operatorname{arctg}(\sqrt{2}-1) = \operatorname{arctg}\dfrac{2(\sqrt{2}-1)}{2\sqrt{2}-2} = \operatorname{arctg} 1 = \dfrac{\pi}{4}$

$\Rightarrow \operatorname{arctg} x_1 = \dfrac{\pi}{8}$.

$\operatorname{arctg} x_2 = \dfrac{\pi}{2} - \operatorname{arctg} x_1 = \dfrac{3\pi}{8}$.

Deci:

$\operatorname{arctg} x_1 \cdot \operatorname{arctg} x_2 = \dfrac{\pi}{8} \cdot \dfrac{3\pi}{8} = \dfrac{3\pi^2}{64}$ $\boxed{C}$

602. Valorile minimă m și maximă M ale expresiei $E(x) = \cos^2 x - 4\sin x$, $x \in \mathbb{R}$ sunt:

(R)

$$E(x) = \cos^2 x - 4\sin x = 1 - \sin^2 x - 4\sin x =$$
$$= -(\sin x + 2)^2 + 5$$

$$1 \le \sin x + 2 \le 3 \,\Big|^2$$

$$1 \le (\sin x + 2)^2 \le 9 \Rightarrow$$

$$-4 \le E(x) \le 4. \Rightarrow m = -4 \text{ și } M = 4 \quad \boxed{D}$$

605. Dacă $x \in \mathbb{R}$ și $\sin(\pi \cos x) = \cos(\pi \sin x)$, atunci $\cos 4x$ este:

(R) Avem: $\sin^2(\pi \cos x) = \cos^2(\pi \sin x)$.

Din formula $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$ avem

$$\sin^2(\pi \cos x) = \frac{1 - \cos(2\pi \cos x)}{2} \text{ și}$$

$$\cos^2(\pi \sin x) = \frac{\cos(2\pi \sin x) + 1}{2}. \text{ Deci:}$$

$$\cos(2\pi \sin x) + \cos(2\pi \cos x) = 0 \Leftrightarrow$$

$$2\cos\big(\pi(\sin x + \cos x)\big)\cos\big(\pi(\sin x - \cos x)\big) = 0$$

Deci $\cos x + \sin x \in \left\{ \frac{2k+1}{2} \,\middle|\, k \in \mathbb{Z} \right\}$ sau

$$\cos x - \sin x \in \left\{ \frac{2k+1}{2} \,\middle|\, k \in \mathbb{Z} \right\}.$$

În același timp

$$\cos x \pm \sin x = \frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}\cos x \pm \frac{\sqrt{2}}{2}\sin x\right) = \sqrt{2}\cos\left(x \mp \frac{\pi}{4}\right) \Rightarrow$$

$\cos x \pm \sin x \in [-\sqrt{2}, \sqrt{2}]$.

Deci $\cos x + \sin x \in \left\{ \frac{2k+1}{2} \mid k \in \mathbb{Z} \right\} \cap [-\sqrt{2}, \sqrt{2}] \Rightarrow$

$\cos x + \sin x \in \left\{ -\frac{1}{2}, \frac{1}{2} \right\}$. Analog deducem:

$\cos x - \sin x \in \left\{ -\frac{1}{2}, \frac{1}{2} \right\}$. Deci

$(\cos x \pm \sin x)^2 = \frac{1}{4} \Leftrightarrow \pm \sin 2x = -\frac{3}{4} \Leftrightarrow$

$\sin^2 2x = \frac{9}{16}$.

$\cos 4x = 1 - 2\sin^2 2x = 1 - \frac{9}{8} = -\frac{1}{8}$ $\boxed{A}$

617. Mulțimea soluțiilor ecuației

$$\left(\sqrt{\frac{1-\sin x}{1+\sin x}} - \sqrt{\frac{1+\sin x}{1-\sin x}}\right)\left(\sqrt{\frac{1-\cos x}{1+\cos x}} - \sqrt{\frac{1+\cos x}{1-\cos x}}\right) = -4$$

este:

R $\sqrt{\frac{1-\sin x}{1+\sin x}} = \sqrt{\frac{1-\sin^2 x}{(1+\sin x)^2}} = \frac{|\cos x|}{1+\sin x}$. Analog

$\sqrt{\frac{1+\sin x}{1-\sin x}} = \frac{|\cos x|}{1-\sin x}$, $\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{|\sin x|}{1+\cos x}$ și

$\sqrt{\frac{1+\cos x}{1-\cos x}} = \frac{|\sin x|}{1-\cos x}$. Deci, avem

$$|\cos x|\left(\frac{1}{1+\sin x} - \frac{1}{1-\sin x}\right) \cdot |\sin x|\left(\frac{1}{1+\cos x} - \frac{1}{1-\cos x}\right) = -4$$

$\Leftrightarrow |\cos x| \cdot \frac{-2\sin x}{\cos^2 x} \cdot |\sin x| \frac{-2\cos x}{\sin^2 x} = -4 \Leftrightarrow$

$\frac{\sin x \cos x}{|\sin x|\,|\cos x|} = -1 \Leftrightarrow \frac{\sin 2x}{|\sin 2x|} = -1 \Leftrightarrow$

$\sin 2x = -|\sin 2x| \Leftrightarrow \sin 2x \leq 0 \Leftrightarrow$

$2x \in \left((2k-1)\pi, 2k\pi\right] \Leftrightarrow x \in \left((2k-1)\frac{\pi}{2}, k\pi\right]$

$\boxed{C}$

643. Fie $z = \frac{(\sqrt{3}+i)^n}{(\sqrt{3}-i)^m}$, $m, n \in \mathbb{N}$. Să se determine relația dintre m și n astfel încât z să fie real.

Ⓡ $\sqrt{3} \pm i = 2\left(\frac{\sqrt{3}}{2} \pm \frac{1}{2} i\right) = 2\left(\cos\frac{\pi}{6} \pm i \sin\frac{\pi}{6}\right)$

Deci,

$$z = \frac{2^n\left(\cos n\frac{\pi}{6} + i \sin n\frac{\pi}{6}\right)}{2^m\left(\cos m\frac{\pi}{6} - i \sin m\frac{\pi}{6}\right)} =$$

$$= 2^{n-m}\left(\cos n\frac{\pi}{6} + i \sin n\frac{\pi}{6}\right)\left(\cos m\frac{\pi}{6} + i \sin m\frac{\pi}{6}\right)$$

$z \in \mathbb{R} \Leftrightarrow \cos n\frac{\pi}{6} \sin m\frac{\pi}{6} + \sin n\frac{\pi}{6} \cos m\frac{\pi}{6} = 0 \Leftrightarrow$

$\sin (n+m)\frac{\pi}{6} = 0 \Leftrightarrow (n+m)\frac{\pi}{6} = k\pi, k \in \mathbb{Z} \Leftrightarrow$

$n+m = 6k, k \in \mathbb{Z}$, $n, m \in \mathbb{N} \Rightarrow$

$n+m = 6k, k \in \mathbb{N}$. $\boxed{E}$

648. Se consideră numerele complexe

$z_1 = \sin a - \cos a + i(\sin a + \cos a)$

$z_2 = \sin a + \cos a + i(\sin a - \cos a)$

Mulțimea valorilor lui a pentru care numărul complex $w = z_1^n + z_2^n$ are modulul maxim este:

Ⓡ Avem:

$$\sin a - \cos a = \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin a - \frac{\sqrt{2}}{2}\cos a\right) =$$

$$= \sqrt{2}\,\sin\left(a - \frac{\pi}{4}\right) = -\sqrt{2}\,\cos\left(a + \frac{\pi}{4}\right)$$

$$\sin a + \cos a = \sqrt{2}\,\sin\left(a + \frac{\pi}{4}\right) = \sqrt{2}\,\cos\left(a - \frac{\pi}{4}\right)$$

Deci:

$$z_1 = -\sqrt{2}\,\cos\left(a + \frac{\pi}{4}\right) + i\,\sqrt{2}\,\sin\left(a + \frac{\pi}{4}\right) =$$

$$= -\sqrt{2}\left(\cos\left(a + \frac{\pi}{4}\right) - i\,\sin\left(a + \frac{\pi}{4}\right)\right)$$

$$z_2 = \sqrt{2}\left(\cos\left(a - \frac{\pi}{4}\right) + i\,\sin\left(a - \frac{\pi}{4}\right)\right).$$

Avem:

$$w = z_1^n + z_2^n = \sqrt{2}^n\left(\left[(-1)^n \cos\left(a + \frac{\pi}{4}\right)\cdot n + \cos\left(a - \frac{\pi}{4}\right)n\right] +\right.$$

$$\left. + i\left[(-1)^{n+1}\sin\left(a + \frac{\pi}{4}\right)n + \sin\left(a - \frac{\pi}{4}\right)n\right]\right)$$

Rezultă

$$|w| = \sqrt{2}^n\sqrt{2 + (-1)^n 2\cos 2an}$$

$$|w| \text{ maxim} \Leftrightarrow (-1)^n \cos 2an = 1 \Rightarrow$$

$$2na \in \{(2k+n)\pi : k \in \mathbb{Z}\} \Rightarrow$$

$$a \in \left\{\frac{k\pi}{n} + \frac{\pi}{2} \,\middle|\, k \in \mathbb{Z}\right\} \quad \boxed{B}$$

649. Mulțimea valorilor lui a pentru care $z_1 + z_2 \in \mathbb{R}$ este:

R) $z_1 + z_2 \in \mathbb{R} \Leftrightarrow \sin\left(a + \frac{\pi}{4}\right) + \sin\left(a - \frac{\pi}{4}\right) = 0 \Leftrightarrow$

$\sin\left(a+\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}-a\right) \iff$

$\frac{\pi}{4}-a = a+\frac{\pi}{4}+2k\pi, k\in\mathbb{Z} \iff a = k\pi, k\in\mathbb{Z},$ $\boxed{A}$

650. Valorile lui n pentru care $z_1^n z_2^n$, $n\in\mathbb{N}^*$, este real și pozitiv sunt:

(R) $z_1^n \cdot z_2^n = (-1)^n \cdot 2^n \left(\cos n\left(a+\frac{\pi}{4}\right) + i \sin n\left(a+\frac{\pi}{4}\right)\right)\left(\cos\left(n\left(a-\frac{\pi}{4}\right)\right) + i \sin n\left(a-\frac{\pi}{4}\right)\right)$

$= (-2)^n\left[\cos n\frac{\pi}{2} - i \sin n\frac{\pi}{2}\right] \in \mathbb{R}_+ \Rightarrow$

$\begin{cases} \sin n\frac{\pi}{2} = 0 \Rightarrow n = 2k, k\in\mathbb{Z} \\ (-2)^n \cos n\frac{\pi}{2} \geq 0 \end{cases}$

$\Rightarrow \cos k\pi \geq 0 \Rightarrow k = 2p \Rightarrow n = 4p, p\in\mathbb{Z}.$

Dar $n\in\mathbb{N}^* \Rightarrow n = 4k, k\in\mathbb{N}^*$ $\boxed{B}$.

651-653 Pentru n și k numere naturale nenule cu n fixat, notăm $a_k = \cos\frac{2k\pi}{n} - 2 + i \sin\frac{2k\pi}{n}$.

Valoarea $\overline{a_n}$ este:

(R) $a_n = \cos 2\pi - 2 + i \sin 2\pi = -1 = \overline{a_n}$ $\boxed{C}$

Valoarea sumei $a_1 + a_2 + \dots + a_n$, $n > 1$ este:

(R) Observăm că $z_k = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}$, $k=\overline{1,n}$ sunt rădăcinile ecuației

$z^n - 1 = 0.$

Deci $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(-2+z_k) = -2n + 0 = -2n.$

$\boxed{A}$

Valoarea produsului $a_1 a_2 \cdot \dots \cdot a_n$ este:

Ⓡ Notăm $S_1 = z_1 + \dots + z_n$

$S_2 = z_1 z_2 + \dots + z_1 z_n + \dots + z_{n-1} z_n$

$\vdots$

$S_p = z_1 z_2 \dots z_p + \dots + z_{n-p+1} z_{n-p+2} \dots z_n$

sumele Viète.

Atunci din ecuația $z^n - 1 = 0$ obținem

$S_1 = 0,\ S_2 = 0, \dots, S_p = 0,\ p < n$ și

$S_n = z_1 \dots z_n = (-1)^n \cdot \frac{-1}{1} = (-1)^{n+1}$.

Avem $\prod_{k=1}^{n} a_k = \prod_{k=1}^{n} (z_k - 2) = (-2)^n + (-2)^{n-1} S_1 + (-2)^{n-2} S_2 + \dots$

$+ (-2)^{n-p} S_p + \dots + (-2) S_{n-1} + S_n = (-2)^n + (-1)^{n+1} =$

$= (-1)^n (2^n - 1)$ $\boxed{D}$.

654. Să se calculeze expresia $E = (\sqrt{3} - i)^8 (-1 + i\sqrt{3})^{11}$.

Ⓡ Deoarece $z_1 = \sqrt{3} - i$ se află în cadranul IV

avem $\arg z_1 = 2\pi - \operatorname{arctg} \frac{1}{\sqrt{3}} = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

$|z_1| = \sqrt{3+1} = 2$, deci $z_1 = 2\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$.

$z_2 = -1 + i\sqrt{3}$ se află în cadranul II $\Rightarrow$

$\arg z_2 = \pi - \operatorname{arctg} \sqrt{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$|z_2| = \sqrt{1+3} = 2 \Rightarrow z_2 = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$.

$E = z_1^8 z_2^{11} = 2^8 \left(\cos \frac{44\pi}{3} + i \sin \frac{44\pi}{3}\right) \cdot 2^{11} \left(\cos \frac{22\pi}{3} + i \sin \frac{22\pi}{3}\right)$

$= 2^{19} \left(\cos \frac{66\pi}{3} + i \sin \frac{66\pi}{3}\right) = 2^{19}$ $\boxed{B}$.

655. Dacă $z+\frac{1}{z}=2\cos\alpha$, atunci expresia $E=z^n+\frac{1}{z^n}$ are, pentru orice $n\in\mathbb{Z}$ şi pentru orice $\alpha\in\mathbb{R}$ valoarea:

(R) Deoarece $z^n+\frac{1}{z^n}=z^{-n}+\frac{1}{z^{-n}}$ $\forall n\in\mathbb{Z}$, putem presupune că $n\in\mathbb{N}$.

Avem $z^2+\frac{1}{z^2}=\left(z+\frac{1}{z}\right)^2-2=4\cos^2\alpha-2=2(2\cos^2\alpha-1)=$

$=2\cos 2\alpha$.

Presupunem că $z^n+\frac{1}{z^n}=2\cos n\alpha$ $\forall n\geq 0$ şi arătăm că $z^{n+1}+\frac{1}{z^{n+1}}=2\cos(n+1)\alpha$.

Într-adevăr

$$z^{n+1}+\frac{1}{z^{n+1}}=\left(z^n+\frac{1}{z^n}\right)\left(z+\frac{1}{z}\right)-\left(z^{n-1}+\frac{1}{z^{n-1}}\right)=$$

$$=2\cos n\alpha\cdot 2\cos\alpha-2\cos(n-1)\alpha=$$

$$=2\left(2\cos n\alpha\cos\alpha-\left(\cos n\alpha\cos\alpha+\sin n\alpha\sin\alpha\right)\right)$$

$$=2\cos(n+1)\alpha.$$

Deci, $E=2\cos n\alpha$, $\forall n\in\mathbb{Z}$. $\boxed{D}$.

658. Fie numărul complex $z=\left(\frac{\sqrt{3}-i}{1+i}\right)^{12}$.

Este adevărată afirmaţia:

(R) $$z=\frac{(\sqrt{3}-i)^{12}(1-i)^{12}}{2^{12}}=\frac{\left[2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)\cdot\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)\right]^{12}}{2^{12}}$$

deoarece $\sqrt{3}-i=2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$ şi

$1-i=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$.

Deci $z = 2^6(\cos 43\pi + i\sin 43\pi) = 2^6(\cos\pi + i\sin\pi)$

$\Rightarrow \arg z = \pi$ $\boxed{B}$.

626. Mulțimea tuturor parametrilor real m pentru care ecuația $\cos^2 x + (m+1)\sin x = 2m-1$ are soluții este:

R) Avem $\cos^2 x = 1 - \sin^2 x$ și notăm

$t = \sin x \in [-1, 1]$.

Ecuația devine:

$$t^2 - (m+1)t + 2(m-1) = 0$$

$$\Delta = (m+1)^2 - 8(m-1) = m^2 - 6m + 9 = (m-3)^2 \geq 0$$

$$t_{1,2} = \frac{m+1 \pm |m-3|}{2}$$

Dacă $m \geq 3 \Rightarrow t_{1,2} = \dfrac{(m+1) \pm (m-3)}{2} = \begin{cases} m-1 \geq 2 \\ 2 \end{cases}$.

Dar vrem rădăcini în $[-1, 1]$, nu ne convine.

Dacă $m < 3 \Rightarrow t_{1,2} = \dfrac{(m+1) \pm (3-m)}{2} = \begin{cases} 2 \\ m-1 < 2 \end{cases}$.

Deci, $m-1 \in [-1, 1] \Leftrightarrow m \in [0, 2]$ $\boxed{A}$.

627. Ecuația $\sin^6 x + \cos^6 x = m$, $m \in \mathbb{R}$ are soluții dacă și numai dacă:

R) $\sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3 =$

$= \underbrace{(\sin^2 x + \cos^2 x)}_{=1} (\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x) =$

$= \left((\sin^2 x + \cos^2 x)^2 - 3 \sin^2 x \cos^2 x \right) = 1 - 3 \sin^2 x \cos^2 x =$

$= 1 - 3 (1 - \cos^2 x) \cdot (\cos^2 x)$.

Notăm $\cos^2 x = t \in [0, 1]$. Deci avem

$1 - 3(1 - t) t = m \Leftrightarrow$

$3t^2 - 3t + 1 - m = 0$

$\Delta = 9 - 12(1 - m) = 12m - 3 \geq 0 \Rightarrow \boxed{m \geq \frac{1}{4}}$

$t_{1,2} = \frac{3 \pm \sqrt{12m - 3}}{6} \in [0, 1] \Leftrightarrow$

$\sqrt{12m - 3} \in [0, 3] \Leftrightarrow m \in \left[\frac{1}{4}, 1\right]$ $\boxed{B}$.